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A SECOND REPORT ON FUNCTIONS FROM THE STATISTICAL  
THEORY OF RESIDUAL CURRENTS IN TIDAL AREAS

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A second report on functions from the statistical theory of residual currents in tidal areas

by

N.M. Temme

#### ABSTRACT

Asymptotic approximations are given for functions occurring in a mathematical model of vorticity production by tidal currents. In an earlier report functions of the same type were considered.

KEY WORDS & PHRASES: *asymptotic expansions of Laplace type integrals, Bessel functions, tidal currents.*



## 1. INTRODUCTION

In an earlier report [2] we considered integrals occurring in a mathematical model of residual circulations by tidal currents. In this report we consider a simpler model, which enables us to obtain more information about some quantities than was possible in the previous case. A central point there was a first order differential equation<sup>\*)</sup>

$$\frac{d\eta(\vec{k}, t)}{dt} - [ik_1 \sin t - b(k)]\eta(\vec{k}, t) = a(\vec{k}) \sin t,$$

where  $\vec{k} = (k_1, k_2)$  is the wave number in a Fourier analysis,  $k = (k_1^2 + k_2^2)^{1/2}$ . The above equation arose in a model for a one-directional tidal current. The simplification concerns the assumption that the current velocity vector rotates circularly. It yields the equation (with  $k_1 = k \cos \theta$ ,  $k_2 = k \sin \theta$ )

$$(1.1) \quad \frac{d\eta(\vec{k}, t)}{dt} - [ik \sin(t+\theta) - b(k)]\eta(\vec{k}, t) = a(\vec{k}) [\tau_1 \sin(t+\theta) + \tau_2 \cos(t+\theta)].$$

The function  $b$  is given by

$$(1.2) \quad b(k) = \tau_2 + \tau_3 k^2,$$

$\tau_1$ ,  $\tau_2$  and  $\tau_3$  are non-negative constants and  $a(\vec{k})$  is related to a stochastic forcing field, the statistics of which are prescribed.

The nontransient solution of the differential equation is written as a Fourier series

$$(1.3) \quad \eta(\vec{k}, t) = \sum_{\ell=-\infty}^{\infty} c_{\ell}(\vec{k}) e^{i\ell(t+\theta)}$$

with

$$(1.4) \quad c_{\ell}(\vec{k}) = i(-i)^{\ell} a(\vec{k}) \sum_{n=-\infty}^{\infty} \frac{J_{n+\ell}(k)}{-b+in} \{ \tau_1 i \frac{n}{k} J_n(k) + \tau_2 J'_n(k) \},$$

where  $J_n(k)$  is the ordinary Bessel function. In order to describe the  $c_{\ell}(\vec{k})$

<sup>\*)</sup> in the above mentioned report the sign of  $b(k)$  was in error

it is convenient to introduce the functions

$$(1.5) \quad R_m(\alpha, k) = \sum_{n=-\infty}^{\infty} \frac{J_n(k) J_{n+m}(k)}{n+\alpha}, \quad m \in \mathbb{Z}, \quad \alpha \notin \mathbb{Z}.$$

With respect to the parameters  $\tau_i$  two cases are investigated, which will be treated in the following sections. First we give some properties of the function  $R_m$  defined above.

In the previous report we used  $R_0(\alpha, t)$ . Explicitly we have (for a proof see [2])

$$(1.6) \quad R_0(\alpha, k) = \frac{\pi}{\sin \pi \alpha} J_{\alpha}(k) J_{-\alpha}(k).$$

By using the well-known recurrence relations

$$(1.7) \quad J_{n \pm 1}(k) = \frac{n}{k} J_n(k) \mp J'_n(k)$$

we obtain

$$(1.8) \quad R_{\pm 1}(\alpha, k) = \frac{1}{k} [1 - \alpha R_0(\alpha, k)] \mp \frac{1}{2} \frac{\partial}{\partial k} R_0(\alpha, k).$$

For general  $m$  we have

$$(1.9) \quad R_{-m}(\alpha, k) = \sum_{n=-\infty}^{\infty} \frac{J_{n-m}(k) J_n(k)}{n+\alpha} = \sum_{n=-\infty}^{\infty} \frac{J_n(k) J_{n+m}(k)}{n+\alpha+m} = R_m(\alpha+m, k).$$

Otherwise we have by using  $J_{-n}(k) = (-1)^n J_n(k)$

$$(1.10) \quad \begin{aligned} R_{-m}(\alpha, k) &= \sum_{n=-\infty}^{\infty} \frac{J_{n-m}(k) J_n(k)}{n+\alpha} = \sum_{n=-\infty}^{\infty} \frac{J_{-n-m}(k) J_{-n}(k)}{-n+\alpha} = \\ &= (-1)^{m+1} R_m(-\alpha, k), \end{aligned}$$

and finally, by using

$$(1.11) \quad J_{n+1}(k) + J_{n-1}(k) = \frac{2n}{k} J_n(k),$$

$$\begin{aligned}
R_{m+1}(\alpha, k) &= \frac{2}{k} \sum_{n=-\infty}^{\infty} \frac{(n+m) J_{n+m}(k) J_n(k)}{n+\alpha} - \sum_{n=-\infty}^{\infty} \frac{J_{n+m-1}(k) J_n(k)}{n+\alpha} = \\
&= -R_{m-1}(\alpha, k) + \frac{2(m-\alpha)}{k} R_m(\alpha, k) + \frac{2}{k} T_m(\alpha, k),
\end{aligned}$$

where

$$T_m(\alpha, k) = \sum_{n=-\infty}^{\infty} J_{n+m}(k) J_n(k).$$

It is known that  $T_0(\alpha, k) = 1$ . For general  $m \in \mathbb{Z}$  we have

$$\begin{aligned}
(1.12) \quad T_m(\alpha, k) &= \sum_{n=-\infty}^{\infty} J_n(k) \frac{1}{2\pi} \int_0^{2\pi} e^{iks \sin \theta - i(n+m)\theta} d\theta = \\
&= \frac{1}{2\pi} \int_0^{2\pi} \sum_{n=-\infty}^{\infty} J_n(k) e^{-in\theta} e^{iks \sin \theta - im\theta} d\theta = \\
&= \frac{1}{2\pi} \int_0^{2\pi} e^{-im\theta} d\theta = \delta_{0,m}
\end{aligned}$$

where Kronecker's symbol  $\delta_{\ell, m}$  is used, defined as

$$(1.13) \quad \delta_{\ell, m} = \begin{cases} 0 & \ell \neq m \\ 1 & \ell = m \end{cases}.$$

Consequently, we have the recursion for  $m \in \mathbb{Z}$

$$(1.14) \quad R_{m+1}(\alpha, k) + R_{m-1}(\alpha, k) = \frac{2(m-\alpha)}{k} R_m(\alpha, k) + \frac{2}{k} \delta_{0,m}.$$

Remark that (1.8) leads to (1.14) with  $m = 0$ . This recursion may be used for the numerical evaluation of  $R_m$  when the two consecutive values are known.

It is interesting to note that (1.14) resembles the recursion relation

$$y_{m+1} + y_{m-1} = \frac{2(m-\alpha)}{x} y_m$$

with solutions  $y_m = J_{m-\alpha}(x)$  and  $y_m = Y_{m-\alpha}(x)$ . This recursion is homogeneous, whereas (1.14) is inhomogeneous.

## 2. THE CASE $\tau_1 = 1$ , $\tau_3 = 0$ , $\tau_2$ VARIABLE ( $\geq 0$ )

In this case we have from (1.2)  $b = \tau_2$ . We use for convenience  $b$  instead of  $\tau_2$  in the notation. We are interested in the two parts of  $c_\ell(k)$  given in (1.4), i.e., we define

$$c_\ell^{(1)}(\vec{k}) = i(-i)^\ell a(\vec{k}) \sum_{n=-\infty}^{\infty} \frac{i n J_{n+\ell}(k) J_n(k)}{k(-b+in)}, \quad (2.1)$$

$$c_\ell^{(2)}(\vec{k}) = -ib(-i)^\ell a(\vec{k}) \sum_{n=-\infty}^{\infty} \frac{J_{n+\ell}(k) J_n'(k)}{-b+in}$$

Especially we want to know the behaviour of  $c_\ell^{(1)} c_\ell^{(1)*}$  and  $c_\ell^{(2)} c_\ell^{(2)*}$  with respect to the parameter  $b$ . The density function  $a(\vec{k})$  does not depend on  $b$  and will not be considered here. In fact we discuss the functions

$$G_\ell^{(1)}(k, b) = \left| \sum_{n=-\infty}^{\infty} \frac{n J_{n+\ell}(k) J_n(k)}{k(-b+in)} \right|^2 \quad (2.2)$$

$$G_\ell^{(2)}(k, b) = b^2 \left| \sum_{n=-\infty}^{\infty} \frac{J_{n+\ell}(k) J_n'(k)}{-b+in} \right|^2$$

and we give the asymptotic behaviour of  $G_\ell^{(1)}(k, b)$  and  $G_\ell^{(2)}(k, b)$  for  $b \rightarrow 0$  and for  $b \rightarrow \infty$  for fixed values of  $k$  and  $\ell$  ( $\ell = 0, 1, 2, 3$ ).

The functions (2.1) give insight in the difference in behaviour of frictional and rotational forcing of vorticity, as the former forcing mechanism is also the major dissipative agency.

### 2.1. Summary of the results

For  $b \rightarrow 0$  we have



$$(2.3) \quad G_{\ell}^{(1)}(k) = \frac{1}{k^2} \{ \delta_{\ell,0} - J_0(k) J_{\ell}(k) \}^2 + O(b),$$

$$G_{\ell}^{(2)}(k) = J_1^2(k) J_{\ell}^2(k) + O(b).$$

and for  $b \rightarrow \infty$  the behaviour is as follows

$$(2.4) \quad \begin{aligned} G_0^{(1)}(k) &= k^2 / (4b^4) + O(b^{-6}) \\ G_{\ell}^{(1)}(k) &= k^{2(\ell-1)} 4^{-\ell} b^{-2\ell} + O(b^{-2\ell-2}), \quad \ell = 1, 2, 3. \end{aligned}$$

$$(2.5) \quad \begin{aligned} G_0^{(2)}(k) &= k^2 b^{-4} + O(b^{-6}) \\ G_{\ell}^{(2)}(k) &= k^{2(\ell-1)} 4^{-\ell} b^{-2(\ell-1)} + O(b^{-2\ell}), \quad \ell = 1, 2, 3. \end{aligned}$$

## 2.2. A further analysis

From (1.5), (1.7), (1.11) and (2.2) it follows that

$$(2.6) \quad \begin{aligned} G_{\ell}^{(1)}(k) &= \frac{1}{k^2} | \delta_{\ell,0} - ib R_{\ell}(ib, k) |^2, \\ G_{\ell}^{(2)}(k) &= \frac{1}{4} b^2 | R_{\ell+1}(1+ib, k) - R_{\ell-1}(-1+ib, k) |^2. \end{aligned}$$

For  $b \rightarrow 0$  we have

$$ib R_{\ell}(ib, k) = J_0(k) J_{\ell}(k) + O(b),$$

$$b R_{\ell+1}(1+ib, k) = -J_1(k) J_{\ell}(k) + O(b),$$

$$b R_{\ell-1}(-1+ib, k) = J_1(k) J_{\ell}(k) + O(b).$$

From these results (2.3) easily follows.

Next we consider the case  $b \rightarrow \infty$ . This is more complicate.

We have

$$(2.7) \quad ib R_{\ell}(-ib, k) = - \sum_{n=-\infty}^{\infty} J_n(k) J_{n+\ell}(k) \frac{1}{1-n/ib}.$$

From

$$\frac{1}{1-n/ib} = \sum_{j=0}^N (n/ib)^j + \frac{(n/ib)^{N+1}}{1-n/ib}$$

one obtains

$$ib R_{\ell}(-ib, k) = - \sum_{j=0}^N \frac{A_j(k; \ell)}{(ib)^j} + B_N$$

with

$$(2.8) \quad A_j(k; \ell) = \sum_{n=-\infty}^{\infty} n^j J_n(k) J_{n+\ell}(k)$$

and

$$B_N = - \sum_{n=-\infty}^{\infty} J_n(k) J_{n+\ell}(k) \frac{(n/ib)^{N+1}}{1-n/ib}$$

Since

$$\left| \frac{1}{1-n/ib} \right| \leq 1, \quad n = 0, \pm 1, \pm 2, \dots, \quad b \in \mathbb{R}$$

we have  $B_N = O(b^{-N-1})$ ,  $b \rightarrow \infty$ . Hence we have the asymptotic expansion

$$(2.9) \quad ib R_{\ell}(-ib, k) \sim - \sum_{j=0}^{\infty} \frac{A_j(k; \ell)}{(ib)^j}, \quad b \rightarrow \infty.$$

It remains to compute the quantities  $A_j(k; \ell)$  defined in (2.8).

From (1.12) we have

$$A_0(k; \ell) = \delta_{\ell, 0}$$

$$A_1(k; \ell) = \frac{k}{2} \sum_{n=-\infty}^{\infty} J_{n+\ell}(k) [J_{n-1}(k) + J_{n+1}(k)] = \frac{k}{2} (\delta_{\ell+1, 0} + \delta_{\ell-1, 0}),$$

where the recursion (1.11) is used. In a similar way we obtain

$$\begin{aligned}
 A_2(k; \ell) &= \frac{k}{2} \{ \sum J_{n+\ell}(k) n J_{n-1}(k) + \sum J_{n+\ell}(k) n J_{n+1}(k) \} = \\
 &= \frac{k}{2} \{ \sum J_{n+\ell}(k) (n-1) J_{n-1}(k) + \sum J_{n+\ell}(k) (n+1) J_{n+1}(k) \\
 &\quad + \sum J_{n+\ell}(k) J_{n-1}(k) - \sum J_{n+\ell}(k) J_{n+1}(k) \} \\
 &= \frac{k}{2} \{ A_1(k; \ell+1) + A_1(k; \ell-1) + A_0(k; \ell+1) - A_0(k; \ell-1) \} \\
 &= \frac{k^2}{4} \{ \delta_{\ell+2,0} + 2\delta_{\ell,0} + \delta_{\ell-2,0} \} + \frac{k}{2} \{ \delta_{\ell+1,0} - \delta_{\ell-2,0} \}.
 \end{aligned}$$

Generalizing this method we obtain the recursion

$$\begin{aligned}
 A_{j+1}(k; \ell) &= \frac{k}{2} \{ \sum J_{n+\ell}(k) n^j J_{n+1}(k) + \sum J_{n+\ell}(k) n^j J_{n-1}(k) \} \\
 &= \frac{k}{2} \{ \sum J_{n+\ell}(k) J_{n+1}(k) \sum_{m=0}^j (-1)^{j-m} (n+1)^m \binom{j}{m} + \\
 &\quad + \sum J_{n+\ell}(k) J_{n-1}(k) \sum_{m=0}^j (n-1)^m \binom{j}{m} \} \\
 &= \frac{k}{2} \sum_{m=0}^j \binom{j}{m} \{ A_m(k; \ell+1) + (-1)^{j-m} A_m(k; \ell-1) \}.
 \end{aligned}$$

For the first order terms in the asymptotic expansion of the functions defined in (2.6) we need

$$\begin{aligned}
 A_1(k; 1) &= k/2 \\
 A_2(k; 0) &= k^2/2, \quad A_2(k; 2) = k^2/4, \\
 A_3(k; 3) &= k^3/8, \\
 A_1(k; 0) &= A_1(k; 2) = A_1(k; 3) = A_2(k; 3) = 0.
 \end{aligned}$$

For  $G_{\ell}^{(1)}(k)$  we obtain

$$G_{\ell}^{(1)}(k) \sim \frac{1}{k^2} \left| \sum_{j=1}^{\infty} \frac{A_j(k; \ell)}{(-ib)^j} \right|^2,$$

and substituting the above values of  $A_j(k; \ell)$  we obtain (2.4). For  $G_{\ell}^{(2)}(k)$  we have for  $b \rightarrow \infty$

$$G_{\ell}^{(2)}(k) \sim \frac{1}{4} b^2 \left| \frac{1}{1+ib} \sum_{j=0}^{\infty} \frac{A_j(k; \ell+1)}{(-1-ib)^j} + \frac{1}{1-ib} \sum_{j=0}^{\infty} \frac{A_j(k; \ell-1)}{(1-ib)^j} \right|^2$$

from which (2.5) easily follows.

### 3. THE CASE $\tau_1 = \tau_2 = 1$ , $\tau_3$ VARIABLE ( $\geq 0$ )

In this case we are interested in the integrals

$$(3.1) \quad \int_0^{\infty} \int_0^{2\pi} k^{\alpha} \langle c_m(\vec{k}) c_{\ell}(\vec{k}) \rangle dk d\theta$$

where  $\langle \rangle$  represents an ensemble average over the stochastic function  $a(\vec{k})$  contained in  $c_{\ell}(\vec{k})$ . For more details on these points we refer to Zimmerman's investigations in [3,4]. In the numerical treatment several density functions  $a(\vec{k})$  were considered. Here we give the formulas for the case of a Gaussian distribution.

The integration with respect to  $\theta$  in (3.1) is trivial due to the circular symmetry and due to the assumed isotropic statistics of  $a(\vec{k})$ . As a consequence, the quantities (3.1) are zero for  $\ell \neq m$ . For  $\alpha = 1$ ,  $\ell = m = 0$ , (3.1) is the residual enstrophy (also considered in [2]), for  $\ell = m = 1$  it is the tidal enstrophy (first harmonic), for  $\ell = m = 2$  it is the tidal enstrophy (second harmonic), etc.

The integrand of (3.1) is expressed in terms of the function

$$(3.2) \quad G_{\ell}(k) = \phi_{\ell}(k) \phi_{\ell}^*(k)$$

where  $*$  means the complex conjugate and

$$(3.3) \quad \phi_{\ell}(k) = \frac{i^{\ell}}{\sqrt{2}} [e^{i\pi/4} R_{1-\ell}(1-ib, k) + e^{-i\pi/4} R_{-1-\ell}(-1-ib, k)].$$

By specifying  $a(\vec{k})$  in (1.4) we write (3.1) as (with different  $\alpha$ )

$$(3.4) \quad H_{\ell}^{(\alpha)}(\lambda) = \int_0^{\infty} k^{\alpha} e^{-k^2 \lambda^{-2}} G_{\ell}(k) dk.$$

Then the enstrophy and the energy are given by

$$(3.5) \quad g_{\ell}(\lambda) = \lambda^{-6} H_{\ell}^{(5)}(\lambda), \quad f_{\ell}(\lambda) = 2\lambda^{-4} H_{\ell}^{(3)}(\lambda)$$

Also the following functions are considered; they all have a physical interpretation

$$(3.6) \quad \begin{aligned} h_{\ell}(\lambda) &= \lambda H_{\ell}^{(2)}(\lambda) / H_{\ell}^{(3)}(\lambda), \\ j_{\ell}(\lambda) &= \left[ \frac{1}{2} f_{\ell}(\lambda) / g_{\ell}(\lambda) \right]^{\frac{1}{2}}, \\ d_{\ell}(\lambda) &= h_{\ell}(\lambda) - j_{\ell}(\lambda). \end{aligned}$$

These functions are computed for  $\ell = 0, 1, 2, 3$  and  $\tau_3 = 0, 10^{-5}, 10^{-4}, 10^{-3}, 10^{-2}, 10^{-1}$ , and  $\lambda \in [10^{-2}, 10^2]$ . The function  $G_{\ell}(k)$  is computed for the same  $\ell$  and  $\tau_3$  values and for  $k \in [10^{-2}, 10^2]$ .

In the following subsections we give the asymptotic behaviour of  $G_{\ell}(k)$  for  $k \rightarrow 0$  and  $k \rightarrow \infty$  (§3.1) and of  $H_{\ell}^{(\alpha)}(\lambda)$  for  $\lambda \rightarrow 0$  and  $\lambda \rightarrow \infty$  (§3.2) for the appropriate values of  $\alpha$  and  $\ell$ .

### 3.1 Asymptotic behaviour of $G_{\ell}(k)$

#### 3.1.1 Results for $k \rightarrow 0$

From (1.6) it follows that

$$R_0(\alpha, k) = \frac{1}{\alpha} \left( 1 - \frac{k^2}{2(1-\alpha^2)} + \frac{3k^4}{8(1-\alpha^2)(4-\alpha^2)} + \dots \right).$$

Hence, by using (1.8), we have

$$R_1(\alpha, k) = \frac{k}{2\alpha(1-\alpha)} - \frac{3k^3}{8\alpha(1-\alpha^2)(2-\alpha)} + \dots$$

and from the recursion (1.14) or directly from (1.5) we obtain

$$R_2(\alpha, k) = \frac{k^2}{4\alpha(1-\alpha)(2-\alpha)} + \dots$$

$$R_3(\alpha, k) = O(k^3), \quad R_m(\alpha, k) = O(k^m), \quad k \rightarrow 0.$$

With (3.3) we have for  $k \rightarrow 0$

$$\begin{aligned} \phi_0(k) &= \frac{k}{8b(b^2+1)} [4(b+1) - 3(2+b)k^2 + O(k^4)] = \\ &= \frac{k}{2} [1 + O(k^2)] \end{aligned}$$

$$(3.7) \quad \phi_1(k) = \frac{1}{2}(i-1)/(1-ib) + O(k^2)$$

$$\phi_2(k) = \frac{k}{4} (1+i)/[(1-ib)(2-ib)] + O(k^3)$$

$$\phi_3(k) = \frac{1}{8} (1-i)k^2/[(1-ib)(2-ib)(3-ib)]$$

From these results it follows that the asymptotic behaviour of  $G_\ell(k)$  for  $k \rightarrow 0$  is given by

$$\begin{aligned} G_0(k) &= \frac{k^2}{4} + O(k^4), \\ G_1(k) &= \frac{1}{4} + O(k^2), \\ (3.8) \quad G_2(k) &= \frac{k^2}{80} + O(k^4), \\ G_3(k) &= \frac{k^4}{3200} + O(k^6). \end{aligned}$$

Summarizing we denote these relations by

$$(3.9) \quad G_\ell(k) = \gamma_\ell k^{\beta_\ell} + O(k^{\beta_\ell+2}), \quad k \rightarrow 0$$

where  $\gamma_\ell$  and  $\beta_\ell$  are readily obtained from (3.8).

### 3.1.2 Results for $k \rightarrow \infty$ .

It is necessary to distinguish between the cases  $\tau_3 > 0$  and  $\tau_3 = 0$ . The first case is simpler than the second one, therefore we begin with it.

#### 3.1.2.1 $\tau_3 > 0$

Recall that  $b = 1 + \tau_3 k^2$ , hence  $b \sim \tau_3 k^2$ . The expansions in (3.7) of  $\phi_\ell(k)$  are also valid for  $k \rightarrow \infty$  (with other  $O$ -terms, however). Explicitly we have for  $k \rightarrow \infty$

$$\begin{aligned}
 \phi_0(k) &\sim -3/[8\tau_3^2 k], \\
 \phi_1(k) &\sim -\frac{1}{2} (1+i)/[\tau_3 k^2], \\
 \phi_2(k) &\sim -\frac{1}{4} (1+i)/[\tau_3^2 k^3], \\
 \phi_3(k) &\sim -(1+i)/[8\tau_3^3 k^4].
 \end{aligned}
 \tag{3.10}$$

From (3.2) it follows that for  $\tau_3 > 0$  and  $k \rightarrow \infty$

$$\begin{aligned}
 G_0(k) &\sim 9/[64\tau_3^4 k^2], \\
 G_1(k) &\sim 1/[2\tau_3^2 k^4], \\
 G_2(k) &\sim 1/[8\tau_3^4 k^6], \\
 G_3(k) &\sim 1/[32\tau_3^6 k^8].
 \end{aligned}
 \tag{3.11}$$

#### 3.1.2.2. $\tau_3 = 0$

Let us first study the case  $\ell = 0$ . From (3.2), (3.3), (1.2), (1.6) and (1.8) it follows that

$$G_0(k) = [\operatorname{Re} R_1(1-i, k) - \operatorname{Im} R_1(1-i, k)]^2, \quad (3.12)$$

$$R_1(\alpha, k) = \frac{1}{k} \left[ 1 - \frac{\pi\alpha}{\sin \pi\alpha} J_{-\alpha}(k) J_{\alpha}(k) \right] - \frac{1}{2} \frac{\pi\alpha}{\sin \pi\alpha} \frac{d}{dk} [J_{-\alpha}(k) J_{\alpha}(k)]$$

The behaviour of  $G_0(k)$  for  $k \rightarrow \infty$  is found by using asymptotic expansions of the Bessel functions. In the previous subsection ( $\tau_3 > 0$ ) the parameter  $\alpha$  was depending on  $k$ . For instance in (3.3) we have  $\alpha = \pm 1 - i b = \pm 1 - i(1 + \tau_3 k^2)$ . Thus if  $\tau_3 > 0$  the order of the Bessel functions also depend on  $k$ . Here, in (3.12), we have  $\alpha = 1 - i$ , a constant. This is the main difference between the cases  $\tau_3 = 0$  and  $\tau_3 > 0$ .

From well-known results of the theory of Bessel functions we derive

$$J_{-\alpha}(x) J_{\alpha}(x) = \frac{1}{\pi x} [(\cos \alpha\pi + \sin 2x) + O(x^{-1})] \quad (3.13)$$

$$\frac{d}{dx} J_{-\alpha}(x) J_{\alpha}(x) = \frac{2 \cos 2x}{\pi x} + O(x^{-2})$$

where  $\alpha$  is a fixed parameter. From this it follows that we have using (3.12)

$$G_0(k) \sim \frac{1}{k^2} \left( 1 + \frac{\cos 2k}{\sinh \pi} \right)^2, \quad k \rightarrow \infty. \quad (3.14)$$

Comparing this result with the first of (3.11) we see that in the present case  $G_0(k)$  has a damped oscillatory behaviour for large  $k$ .

For very small values of  $\tau_3$  this behaviour also occurs; also in that case, however,  $G_0(k)$  ultimately behaves as in (3.10).

From the recursion in (1.14) and from (1.6) and (1.8) it follows that for  $m \geq 2$

$$R_m = O(k^{-1}), \quad k \rightarrow \infty \quad (3.15)$$

and that  $G_{\ell}(k)$  for  $\ell = 1, 2, \dots$  behaves as  $G_0(k)$ : a damped oscillatory behaviour at infinity. More details on this point will not be given here.



### 3.2 The asymptotic behaviour of $H_{\ell}^{(\alpha)}(\lambda)$ for $\lambda \rightarrow 0$ and $\lambda \rightarrow \infty$

#### 3.2.1 Results for $\lambda \rightarrow 0$

The asymptotic expansion of  $H_{\ell}^{(\alpha)}(\lambda)$  for  $\lambda \rightarrow 0$  is obtained as follows. Standard methods from asymptotic analysis yield using (3.4) and (3.9)

$$H_{\ell}^{(\alpha)}(\lambda) \sim \gamma_{\ell} \int_0^{\infty} k^{\alpha} e^{-k^2 \lambda^{-2} \beta_{\ell}} dk = \frac{1}{2} \gamma_{\ell} \Gamma[(1+\alpha+\beta_{\ell})/2] \lambda^{1+\alpha+\beta_{\ell}}.$$

From this we obtain for the functions in (3.5) and (3.6) the asymptotic behaviour for  $\lambda \rightarrow 0$

$$g_{\ell}(\lambda) \sim \frac{1}{2} \gamma_{\ell} \Gamma(3+\frac{1}{2}\beta_{\ell}) \lambda^{\beta_{\ell}},$$

$$f_{\ell}(\lambda) \sim \gamma_{\ell} \Gamma(2+\frac{1}{2}\beta_{\ell}) \lambda^{\beta_{\ell}},$$

$$h_{\ell}(\lambda) \sim \Gamma(1\frac{1}{2}+\frac{1}{2}\beta_{\ell})/\Gamma(2+\frac{1}{2}\beta_{\ell}),$$

$$j_{\ell}(\lambda) \sim (2 + \frac{1}{2} \beta_{\ell})^{-\frac{1}{2}},$$

$$d_{\ell}(\lambda) = h_{\ell}(\lambda) - j_{\ell}(\lambda).$$

#### 3.2.2 Results for $\lambda \rightarrow \infty$

To obtain the asymptotic expansion of  $H_{\ell}^{(\alpha)}(\lambda)$  for  $\lambda \rightarrow \infty$  it is convenient to write it, by means of Mellin transform technique, as follows

$$(3.16) \quad H_{\ell}^{(\alpha)}(\lambda) = \frac{\lambda^{\alpha+1}}{4\pi i} \int_L \lambda^{-z} \Gamma[(\alpha+1-z)/2] M[G_{\ell}, z] dz,$$

where

$$(3.17) \quad M[G_{\ell}, z] = \int_0^{\infty} k^{z-1} G_{\ell}(k) dk$$

is the Mellin transform of  $G_{\ell}$ . The contour of integration  $L$  in (3.16) is a

vertical, such that the singularities of the gamma function at

$$(3.18) \quad z_j = \alpha + 1 + 2j, \quad j = 0, 1, 2, \dots$$

and the singularities of  $M[G_\ell, z]$  are at the right of it.

For details on this method we refer to Bleistein & Handelsman [1, Ch.4].

The poles of  $M[G_\ell, z]$  can be localized by using the behaviour of  $G_\ell$  for large  $k$ . Again, we have to distinguish between two cases:  $\tau_3 = 0$ ,  $\tau_3 > 0$ .

### 3.2.2.1. $\tau_3 = 0$

In this event we use (3.14). It follows that  $M[G_\ell, z]$  has a pole at  $z = 2$ . (For  $\text{Re } z < 2$  the integral (3.17) converges at infinity). The values of  $\alpha$  to be considered in (3.18) are (see (3.5) and (3.6)) are

$$(3.19) \quad \begin{aligned} \alpha = 2, & \quad \text{giving } z_j = 3 + 2j \\ \alpha = 3, & \quad \text{giving } z_j = 4 + 2j \\ \alpha = 5, & \quad \text{giving } z_j = 6 + 2j \end{aligned}$$

Hence, poles  $z_j$  are in all three cases larger than 2 (the pole of  $M[G_\ell, z]$ ). By replacing the vertical  $L$  in (3.16) to the right we obtain by taking the residue at  $z = 2$

$$H_\ell^{(\alpha)}(\lambda) = C_\ell^{(\alpha)} \lambda^{\alpha-1} + O(\lambda^{\alpha-2}), \quad \lambda \rightarrow \infty,$$

where  $C_\ell^{(\alpha)}$  is a constant, not depending on  $\lambda$ . For the functions in (3.5) and (3.6) we have

$$(3.20) \quad \begin{aligned} g_\ell(\lambda), f_\ell(\lambda) &= O(\lambda^{-2}) \\ h_\ell(\lambda), j_\ell(\lambda), d_\ell(\lambda) &= O(1) \end{aligned} \quad \lambda \rightarrow \infty$$

with  $\tau_3 = 0$

3.2.2.2.  $\tau_3 > 0$ 

The asymptotic behaviour of  $G_\ell(k)$  for  $k \rightarrow \infty$  is given by (3.11). Hence, the poles of  $M[G_\ell, z]$  occur at  $z = 2\ell + 2$  ( $\ell = 0, 1, 2, 3$ ). For  $\ell = 0$  the situation is as in subsection 3.2.2.1 with results in (3.20). For  $\ell = 1$  we have for  $\alpha = 2$  (see (3.19)) a pole  $z_j$  in 3 and a pole of  $M[G_1, z]$  in 4. Hence, the pole 3 gives the main contribution. For  $\ell = 1$ ,  $\alpha = 3$  the integrand of (3.16) has a double pole at  $z = 4$ . As a consequence, the asymptotic behaviour of  $H_1^{(3)}(\lambda) = O(\ell \ln \lambda)$ . Combining all the possible combinations of  $\ell$  and  $\alpha$  we obtain the following table for the asymptotic behaviour of the functions of (3.5), (3.6).

	$\ell = 0$	$\ell = 1$	$\ell = 2$	$\ell = 3$
$g_\ell(\lambda)$	$\lambda^{-2}$	$\lambda^{-4}$	$\lambda^{-6} \ell \ln \lambda$	$\lambda^{-6}$
$f_\ell(\lambda)$	$\lambda^{-2}$	$\lambda^{-4} \ell \ln \lambda$	$\lambda^{-4}$	$\lambda^{-4}$
$h_\ell(\lambda)$	1	$\lambda / \ell \ln \lambda$	$\lambda$	$\lambda$
$j_\ell(\lambda)$	1	$(\ell \ln \lambda)^{\frac{1}{2}}$	$\lambda / (\ell \ln \lambda)^{\frac{1}{2}}$	$\lambda$
$d_\ell(\lambda)$	1	$\lambda / \ell \ln \lambda$	$\lambda$	$\lambda$

## 4. A FURTHER GENERALIZATION

Here we consider some quantities which arise in the study of a generalization of the differential equation (1.1). We use different frequencies in the forcing and damping terms of the equation and we are interested in the analogues of the  $G$  and  $H$  functions of the previous sections, especially in those components in  $G$  and  $H$  depending on the difference of the two frequencies. To be more specific, let us write the differential equation as

$$(4.1) \quad \frac{d\eta(k, t)}{dt} = \{ik(u_1 \sin \sigma_1 t + u_2 \sin \sigma_2 t) - b\sigma_1\} \eta(k, t) + \\ ka(k)\{u_1(\sin \sigma_1 t + \cos \sigma_1 t) + u_2(\sin \sigma_2 t + \cos \sigma_2 t)\},$$

where  $u_1, u_2, \sigma_1, \sigma_2, b$  are real parameters, not depending on  $t$  and  $k$ .

The nontransient solution of this equation is

$$(4.2) \quad \eta(k, t) = \sqrt{2} \, ka(k) e^{-b\sigma_1 t - ik(\cos \sigma_1 t + \gamma \cos \sigma_2 t)} \times \\ \int_0^t e^{b\sigma_1 \tau + ik(\cos \sigma_1 \tau + \gamma \cos \sigma_2 \tau)} [u_1 \cos(\sigma_1 \tau - \frac{\pi}{4}) + \\ u_2 \cos(\sigma_2 \tau - \frac{\pi}{4})] d\tau,$$

where

$$(4.3) \quad \kappa = u_1 k / \sigma_1, \quad \beta_1 = \sigma_1 / \sigma_2, \\ \gamma = \beta_1 \beta_2, \quad \beta_2 = u_2 / u_1.$$

By expanding the exponential functions

$$e^{-ik \cos \sigma_1 t}, e^{-i\gamma \kappa \cos \sigma_2 t}, e^{ik \cos \sigma_1 \tau}, e^{i\gamma \kappa \cos \sigma_2 \tau}$$

as Fourier series, as  $e^{ix \cos t} = \sum_i^m e^{imt} J_n(x)$ , and by integrating the resulting series we obtain

$$(4.4) \quad \eta(k, t) = \frac{\kappa a(k)}{\sqrt{2}} e^{-i\pi/4} \sum_{r, s, n, m} i^{m+n-r-s} J_m(\kappa) J_n(\gamma \kappa) J_r(\kappa) J_s(\gamma \kappa) \times \\ [u_1 \{ \frac{e^{i(n\sigma_2 + (m+1)\sigma_1 + r\sigma_1 + s\sigma_2)t}}{b\sigma_1 + i(m+1)\sigma_1 + in\sigma_2} + \\ + i \frac{e^{i(n\sigma_2 + (m-1)\sigma_1 + r\sigma_1 + s\sigma_2)t}}{b\sigma_1 + i(m-1)\sigma_1 + in\sigma_2} \} \\ + u_2 \{ \frac{e^{i(m\sigma_1 + (n+1)\sigma_2 + r\sigma_1 + s\sigma_2)t}}{b\sigma_1 + im\sigma_2 + i(n+1)\sigma_2} + \\ + i \frac{e^{i(m\sigma_1 + (n-1)\sigma_2 + r\sigma_1 + s\sigma_2)t}}{b\sigma_1 + im\sigma_2 + i(n-1)\sigma_2} \} ].$$

The  $r, s, n, m$  - values run from  $-\infty$  to  $+\infty$ . It is clear that the series (4.4) can be written as

$$(4.5) \quad \sum_{p, q, r, s} a_{p, q, r, s} e^{i(p+r)\sigma_1 t + i(q+s)\sigma_2 t}$$

This series can be split up into

$$\sum_j \alpha_j^{(1)} e^{ij\sigma_1 t} + \sum_j \alpha_j^{(2)} e^{ij\sigma_2 t} + \sum_j \alpha_j^{(3)} e^{ij(\sigma_1 + \sigma_2)t} \\ + \sum_j \alpha_j^{(4)} e^{ij(\sigma_2 - \sigma_1)t} + \dots \text{ (ad infinitum).}$$

Here we are interested in the coefficient  $\alpha_1^{(4)}$ . It arises in (4.5) if  $p + r = -1, q + s = 1$ . Hence

$$\alpha_1^{(4)} = \sum_{p,q} a_{p,q,-J-p,1-q}.$$

When applying this to (4.4), we infer that the Fourier coefficient of  $e^{i(\sigma_2 - \sigma_1)t}$  is given by

$$\alpha_1^{(4)} = ia(k) \phi_{\sigma_2 - \sigma_1}(\kappa)$$

with  $\phi_{\sigma_2 - \sigma_1}$  given by the double sum

$$(4.6) \quad \phi_{\sigma_2 - \sigma_1}(\kappa) = \frac{\kappa e^{i\pi/4}}{\sqrt{2}} \sum_{p,q} \frac{J_{p-1}(\kappa) J_{q+1}(\gamma\kappa)}{b + ip + iq/\beta_1} \{A_{p,q} + \beta_2 B_{p,q}\},$$

with

$$A_{p,q} = J_{p+1}(\kappa) J_q(\gamma\kappa) + iJ_{p-1}(\kappa) J_q(\gamma\kappa),$$

$$B_{p,q} = J_p(\kappa) J_{q+1}(\gamma\kappa) + iJ_p(\kappa) J_{q-1}(\gamma\kappa),$$

and  $\beta_1, \beta_2$  defined in (4.3).

Then we define the function (the analogue of (3.2))

$$G_{\sigma_2 - \sigma_1}(\kappa) = \phi_{\sigma_2 - \sigma_1}(\kappa) \phi_{\sigma_2 - \sigma_1}^*(\kappa)$$

and furthermore

$$(4.5) \quad \psi_{\sigma_2 - \sigma_1}(\kappa) = \text{phase}[\phi_{\sigma_2 - \sigma_1}(\kappa)];$$

the relation between  $\phi$ ,  $G$  and  $\psi$  is

$$\phi_{\sigma_2-\sigma_1}(\kappa) = G_{\sigma_2-\sigma_1}^{\frac{1}{2}}(\kappa) e^{i\psi_{\sigma_2-\sigma_1}(\kappa)},$$

where we take  $\psi_{\sigma_2-\sigma_1}(\kappa) \in (-\pi, \pi]$ . The analogues of (3.4) and (3.5) are also considered. We define

$$H_{\sigma_2-\sigma_1}^{\alpha}(\lambda) = \int_0^{\infty} \kappa^{\alpha} e^{-\kappa^2 \lambda^{-2}} G_{\sigma_2-\sigma_1}(\kappa) d\kappa$$

$$g_{\sigma_2-\sigma_1}(\lambda) = \lambda^{-6} H_{\sigma_2-\sigma_1}^{(5)}(\lambda)$$

$$f_{\sigma_2-\sigma_1}(\lambda) = 2\lambda^{-4} H_{\sigma_2-\sigma_1}^{(3)}(\lambda).$$

The function  $G_{\sigma_2-\sigma_1}(\kappa)$  is computed for

$$\beta_1 = .966, \quad \beta_2 = .1, .4, .7, 1.0, \quad b = 1, \quad k \in [10^{-2}, 10^2],$$

the function  $\psi_{\sigma_2-\sigma_1}(\kappa)$  for

$$\beta_1 = .966, \quad \beta_2 = .4, \quad b = 10^{-2}, 10^{-1}, 1, 10, \quad k \in [10^{-2}, 10^2]$$

and  $g_{\sigma_2-\sigma_1}(\lambda)$ ,  $f_{\sigma_2-\sigma_1}(\lambda)$  for

$$\beta_1 = .966, \quad \beta_2 = .4, \quad b = 1, \quad \lambda \in [10^{-2}, 10^2].$$

## 5. SOME REMARKS ON THE COMPUTATIONS

For the computations of  $G_{\ell}(k)$  we used the representations (3.2) with (3.2) and (1.5). The series in (1.5) converges very fast, since for large  $n$  we have

$$J_n(k) J_{n+m}(k) \sim \frac{(k/2)^{2n+m}}{n! (n+m)!}, \quad m \geq 0;$$

for  $m < 0$  we used (1.10). The  $\lambda$ -integrals  $H_{\ell}^{(\alpha)}(\lambda)$  in (3.4) were computed by using a trapezoidal rule (after a suitable transformation). For details see the previous report [2]. The function of Section 4 are computed as their analogues of Section 3.

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